

LOWER-ENVELOPE RIGIDITY IN GAUGE TRANSPORT: MASS GAPS, INFRARED SOFTNESS, TORSION CHARGE, AND AFFINE OBSERVER RESPONSE

J. SCOTT LITTLE

ABSTRACT. A mass gap and a rigid comparison problem are easy to place too close together. They are not the same statement. A mass gap controls every nonvacuum direction. Here the question is narrower: can a certified nontrivial response be driven arbitrarily close to the point where that comparison becomes degenerate? This gives a lower-envelope notion of rigidity.

Two examples force the distinction. For future-cone momentum p , affine observer response satisfies

$$\inf_u p \cdot u = \sqrt{p^2}.$$

Timelike momentum has a positive lower envelope; nonzero null momentum does not. For a nonzero element ℓ of a finitely generated discrete Abelian unscreened charge group A , the character response $|\alpha(\ell) - 1|^2$ has a positive punctured lower envelope exactly when ℓ is torsion. If $m = \text{ord}(\ell) < \infty$, that reserve is $4 \sin^2(\pi/m)$; infinite-order charge is character-soft. The physics is different. The lower-envelope geometry is the same.

Energy enters only after another theorem. At finite certified endpoints, a response-weighted inequality $K \geq \kappa W$ is weaker than full off-ground coercivity, and its compatibility with soft directions is controlled by a finite Schur complement. Free Maxwell theory is the control case: the one-photon gap vanishes while a positive finite-observer response inequality survives. On the charge side, the selected Dirichlet passage gives a concrete energetic bridge: a charge-twisted lift fails ordinary gluing by the exact factor $\alpha(\ell) - 1$, and the local boundary form measures that defect.

Torsion therefore supplies an isolated charge response, not a mass gap. Degree-three associator data remain coherence data; in the selected $SU(3)$ passage problem they act by unitary reassociation rather than by generating the passage spectrum. The rest of the paper keeps those boundaries explicit while treating reductive gapped/soft coexistence, limit transfer, and residual soft gauge sectors.

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1. INTRODUCTION

1.1. The question is weaker than a mass gap. A mass gap is a strong answer to a strong question. If a physical Hamiltonian $H \geq 0$ has complete-ground projection P_Ω , then

$$(1) \quad H \geq \Delta(1 - P_\Omega), \quad \Delta > 0,$$

forbids every normalized nonvacuum sequence whose energy tends to zero. Nothing soft survives that estimate.

The question here is weaker. Fix a physical sector and a certified family of comparisons. Each comparison reads some nonnegative response. Can the nontrivial response be pushed arbitrarily close to the place where that comparison stops seeing anything? If not, there is a reserve. If yes, the sector is soft for that comparison problem.

This distinction matters because a transport theory carries several comparison problems at once. Observer energy is one. Meridional charge detection is another. They should not be identified merely because both admit a useful infimum. The common object is the lower envelope; the physics sits in the response being enveloped.

1.2. Two places where the same mathematics appears. For future-cone momentum p , the affine response to a future-timelike observer u is

$$R_p(u) = p \cdot u.$$

The observer-complete structure imported from [10] gives

$$\inf_{u \in H_+^3} p \cdot u = \sqrt{p^2}.$$

A timelike momentum therefore carries a positive observer reserve. A nonzero null momentum does not: every finite observer still measures positive energy, but a sequence of boosts can drive that energy toward zero. The response remains meaningful all the way to the boundary. The lower envelope is what changes.

Now take a nonzero unscreened Abelian charge $\ell \in A$. Its comparison family is the Pontryagin dual

$$\hat{A} = \text{Hom}(A, U(1)),$$

and the natural meridional response is

$$r_\ell(\alpha) = |\alpha(\ell) - 1|^2.$$

Here the same question becomes arithmetic. If ℓ has finite order, the nontrivial character values cannot approach 1 continuously. If ℓ has infinite order, they can. Torsion and infinite order therefore split the charge problem into rigid and soft lower-envelope sectors.

That is the point of departure:

rigidity means separation of certified response from degeneracy.

This is not yet an energy theorem.

1.3. The energetic arrow is a separate burden. The passage from a positive response reserve to a physical energy floor is not automatic. It requires an analytic bridge. On the observer side that bridge may be a Hamiltonian or closed form dominating a response functional. On the charge side the selected $SU(3)$ confinement construction [7] gives something more concrete: the charged cut lift satisfies the source-twisted matching law, fails ordinary admissible matching, and the Friedrichs boundary form measures precisely that ordinary defect.

There is a second shortcut to avoid. The finite recognition architecture of [8] keeps relative charge, degree-three coefficient-band associator data, analytic forms, and coercive coordinates separately typed. Thus

$$[\omega_1] \neq [\omega_2] \not\Rightarrow q_1 \neq q_2.$$

In the selected passage problem, the relative center charge supplies the phase defect. The associator supplies coherent reassociation. Those are different jobs.

1.4. Proof architecture and status boundary. The proof order follows the burden. First isolate the lower envelope. Then test it on observers and charges. Only after that do we ask what a physical form does with the response. Maxwell is the control case because any hidden reintroduction of a mass gap would fail there.

Three boundaries are load-bearing. The Moore–Penrose formulas below are finite-endpoint statements. Comparison–Mosco convergence is imported only after it has been independently certified. No lower-envelope reserve is promoted to a Hamiltonian gap without another analytic theorem.

2. LOWER-ENVELOPE RIGIDITY

Definition 2.1 (Certified response system). Let X be a set of nontrivial physical sectors. For every $x \in X$, let $\mathcal{C}_x$ be a certified comparison family and let

$$r_x : \mathcal{C}_x \longrightarrow [0, \infty)$$

be a nonnegative response. Define the punctured response set

$$\mathcal{R}_x^\times = \{r_x(c) : c \in \mathcal{C}_x, r_x(c) > 0\}.$$

Whenever $\mathcal{R}_x^\times$ is nonempty, define

$$(2) \quad \rho(x) = \inf \mathcal{R}_x^\times.$$

For a family of sectors define the outer reserve

$$(3) \quad \rho_*(X) = \inf_{x \in X} \rho(x).$$

Proposition 2.2 (Separation criterion). *For every sector with nonempty punctured response set,*

$$\rho(x) > 0 \iff 0 \notin \overline{\mathcal{R}_x^\times},$$

whereas

$$\rho(x) = 0 \iff 0 \in \overline{\mathcal{R}_x^\times}.$$

Thus lower-envelope rigidity is precisely separation of the punctured response set from the degeneracy value 0.

Proof. This is the defining property of the infimum of a nonempty subset of $(0, \infty)$. $\square$

Theorem 2.3 (Energetic lift of a lower envelope). *Suppose that for every $x \in X$, every certified comparison $c \in \mathcal{C}_x$, and every vector u in the declared represented sector, a nonnegative physical form satisfies*

$$(4) \quad q_{x,c}[u] \geq C_* r_x(c) \|u\|^2$$

with one constant $C_* > 0$. Then every nondegenerate response satisfies

$$q_{x,c}[u] \geq C_* \rho(x) \|u\|^2.$$

If in addition $\rho_*(X) > 0$, then

$$(5) \quad q_{x,c}[u] \geq C_* \rho_*(X) \|u\|^2$$

uniformly over all sectors and all nondegenerate certified comparisons.

Proof. For $r_x(c) > 0$, one has $r_x(c) \geq \rho(x) \geq \rho_*(X)$. Insert these inequalities into (4). $\square$

Remark 2.4 (The bridge is indispensable). Theorem 2.3 runs in one direction. The lower envelope says that a response cannot collapse. It does not say what that response costs. The constant C_* is the missing physics, and it has to be earned on the represented carrier.

3. APPLICATION I: AFFINE OBSERVER RESPONSE

Let $p \in V_+ \setminus \{0\}$ and let H_+^3 denote the future unit hyperboloid. On the affine observer-response structure of [10], define

$$r_p(u) = p \cdot u, \quad u \in H_+^3.$$

The imported affine lower-envelope identity is

$$(6) \quad \rho_{\text{obs}}(p) := \inf_{u \in H_+^3} p \cdot u = \sqrt{p^2}.$$

For timelike p , the reserve is positive. For nonzero null p , every finite observer response remains positive but the infimum is zero, so 0 lies in the closure of the punctured response orbit.

For a realized joint spectral support $\Sigma \subset V_+$, define the outer observer reserve

$$(7) \quad \Delta_{\text{obs}}(\Sigma) = \inf_{p \in \Sigma \setminus \{0\}} \rho_{\text{obs}}(p).$$

Its positivity depends on the realized spectral support and is therefore dynamical. In the strong semisimple situation imported from GAT VI, a positive observer lower envelope participates in invariant mass transfer. The free Maxwell application below shows that the response architecture itself remains meaningful when this outer reserve vanishes.

4. APPLICATION II: TORSION CHARGE AND CHARACTER SOFTNESS

Let A be a finitely generated discrete Abelian unscreened charge group and

$$\widehat{A} = \text{Hom}(A, U(1))$$

its Pontryagin dual. For $\ell \in A \setminus \{0\}$ define

$$(8) \quad r_\ell(\alpha) = |\alpha(\ell) - 1|^2, \quad \delta_A(\ell) = \inf_{\alpha(\ell) \neq 1} r_\ell(\alpha).$$

Theorem 4.1 (Torsion–softness dichotomy). *Let $\ell \in A \setminus \{0\}$. If $m = \text{ord}(\ell) < \infty$, then*

$$(9) \quad \delta_A(\ell) = 4 \sin^2 \frac{\pi}{m} > 0.$$

If ℓ has infinite order, then

$$(10) \quad \delta_A(\ell) = 0.$$

Equivalently, $\delta_A(\ell) > 0$ exactly for nonzero torsion charge.

Proof. Write $A \simeq \mathbb{Z}^r \oplus T$ with T finite. If ℓ has order m , then $\langle \ell \rangle \simeq \mathbb{Z}_m$. A character of this cyclic subgroup sending ℓ to a primitive m th root extends to A because $U(1)$ is divisible. Hence the possible values on ℓ are exactly the m th roots of unity. The nearest nontrivial value to 1 is therefore $e^{\pm 2\pi i/m}$, giving (9).

If ℓ has infinite order, its projection to the free factor is nonzero. Choose an integer linear functional on the free factor that is nonzero on ℓ and compose it with $n \mapsto e^{in\theta}$. Varying θ gives nontrivial character values on ℓ arbitrarily close to 1. Hence (10). $\square$

Corollary 4.2 (Group-wide character reserve). *For nontrivial finitely generated A , define*

$$\delta_*(A) = \inf_{\ell \neq 0} \delta_A(\ell).$$

Then $\delta_*(A) > 0$ if and only if A is finite. If A is finite with exponent $e(A)$, then

$$(11) \quad \delta_*(A) = 4 \sin^2 \frac{\pi}{e(A)}.$$

Proof. If A has nonzero free rank, it contains an infinite-order element, so the infimum vanishes by theorem 4.1. If A is finite, every nonzero element has finite order dividing $e(A)$, and an element of order $e(A)$ exists. Since $4 \sin^2(\pi/m)$ decreases with $m \geq 2$, (11) follows. $\square$

Proposition 4.3 (Choice-free detector average). *If A is finite and $\ell \neq 0$, then*

$$(12) \quad \sum_{\alpha \in \hat{A}} |\alpha(\ell) - 1|^2 = 2|A|.$$

For finite A , the normalized average is always 2 and therefore does not record the order of ℓ . The lower envelope does.

Proof. Character orthogonality gives $\sum_{\alpha \in \hat{A}} \alpha(\ell) = 0$ for $\ell \neq 0$. Expanding $|\alpha(\ell) - 1|^2 = 2 - \alpha(\ell) - \overline{\alpha(\ell)}$ and summing yields (12). $\square$

Remark 4.4 (Screening acts before the envelope). The charge group entering [theorem 4.1](#) is already the screened quotient. If L is the admitted line-charge group and S_{dyn} is generated by inhabited dynamical screening sectors, then

$$A_{\text{uns}} = L/S_{\text{dyn}}.$$

Screening changes the set over which the outer envelope is taken. It can turn infinite order into torsion or erase the sector altogether. That is the algebraic effect. Any confinement, Higgs, or mass statement still needs its own dynamics.

5. IMPORTED INTERFACES AND COORDINATE FIREWALL

Before touching energy, one more firewall. Several coordinates live on the same finite proposal. Shared notation does not make them implications of one another.

Proposition 5.1 (Coordinate independence). *A distinction of realized associator classes does not formally imply a distinction of physical quadratic forms:*

$$[\omega_1] \neq [\omega_2] \not\Rightarrow q_1 \neq q_2.$$

Likewise neither $[\omega] \neq 0$ nor finiteness of $Z(G)$ implies a positive spectral gap.

Status. This boundary is imported. In [\[8\]](#), the internal H^3 coordinate and the electric/coercive coordinate are carried separately, and the witness ledger already contains different coefficient-band sectors with the same finite Hamiltonian and coercive data. There is therefore no missing implication to be supplied here. $\square$

Remark 5.2 (Why the firewall is load-bearing). The later parameter τ is not a norm of a cohomology class. It measures a represented analytic handoff. If those two quantities are identified, the rest of the paper becomes circular.

5.1. Comparison–Mosco boundary. The limit passage is another place where a stronger theorem can be smuggled in by notation. We do not do that. Mosco convergence of closed convex forms begins with [11]; varying-Hilbert-space versions may be found in [6]. The GAT comparison package of [9] is imported only when its hypotheses have been checked.

Section 11 starts after that work. Once $q_n \rightarrow q_0$ in the required comparison–Mosco sense, and once the response admits compatible recovery along the same passage, the response floor transfers. Response coercivity itself is not used to manufacture the convergence theorem.

6. FINITE RESPONSE-WEIGHTED COERCIVITY

Now the finite matrix problem. The Moore–Penrose formulas below belong to a certified endpoint; no continuum pseudoinverse is being hidden in the notation. The block-positivity criterion goes back to [1]; see also [13].

Definition 6.1 (Response pair). Let K, W be Hermitian forms on $\mathcal{V}$ with $K \geq 0$, $W \geq 0$, and $W \neq 0$. Define

$$(13) \quad \kappa(K, W) = \inf_{\langle v, Wv \rangle > 0} \frac{\langle v, Kv \rangle}{\langle v, Wv \rangle}.$$

The pair is *response-coercive* when $\kappa(K, W) > 0$. Equivalently, there exists $\kappa > 0$ such that $K \geq \kappa W$.

Proposition 6.2 (Two-block response criterion). *Let*

$$K = \begin{pmatrix} a & -\eta \\ -\eta & n \end{pmatrix}, \quad W = \begin{pmatrix} r^2 & 0 \\ 0 & s^2 \end{pmatrix},$$

with nonnegative scalar data. Then $K \geq \kappa W$ if and only if

$$(14) \quad a \geq \kappa r^2, \quad n \geq \kappa s^2, \quad \eta^2 \leq (a - \kappa r^2)(n - \kappa s^2).$$

Proof. The inequality is equivalent to positivity of

$$K - \kappa W = \begin{pmatrix} a - \kappa r^2 & -\eta \\ -\eta & n - \kappa s^2 \end{pmatrix}.$$

For a real symmetric 2×2 matrix, positivity is equivalent to nonnegative diagonal entries together with nonnegative determinant. $\square$

Theorem 6.3 (Finite generalized Schur criterion). *Let*

$$K = \begin{pmatrix} A & B^* \\ B & N \end{pmatrix}, \quad W = \begin{pmatrix} R_A & 0 \\ 0 & R_S \end{pmatrix}$$

be finite Hermitian matrices. Fix $\kappa \geq 0$ and define

$$A_\kappa = A - \kappa R_A, \quad N_\kappa = N - \kappa R_S.$$

Then $K \geq \kappa W$ if and only if

$$(15) \quad N_\kappa \geq 0, \quad \text{Ker } N_\kappa \subseteq \text{Ker } B^*, \quad A_\kappa - B^* N_\kappa^\dagger B \geq 0.$$

Proof. Because the endpoint space is finite dimensional, $\text{Ran } N_\kappa$ is closed and the Moore–Penrose inverse is a bounded operator on the supported subspace. Positivity on vectors whose soft component lies in $\text{Ker } N_\kappa$ forces $B^*|_{\text{Ker } N_\kappa} = 0$. On the orthogonal support of N_κ , completing the square gives

$$\begin{aligned} \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, (K - \kappa W) \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \left\langle y + N_\kappa^\dagger Bx, N_\kappa(y + N_\kappa^\dagger Bx) \right\rangle \\ &\quad + \left\langle x, (A_\kappa - B^* N_\kappa^\dagger B) x \right\rangle. \end{aligned}$$

This gives the equivalence. See also [1, 13]. $\square$

Definition 6.4 (Schur screening load). When the hypotheses of [theorem 6.3](#) are met, define

$$\Sigma_\kappa = B^* N_\kappa^\dagger B.$$

Then the response-active residual is

$$S_\kappa = A - \kappa R_A - \Sigma_\kappa.$$

Corollary 6.5 (Soft–rigid scaling). *In the scalar model of [theorem 6.2](#), suppose $n_R \rightarrow 0$ and the active response reserve stays uniformly bounded,*

$$0 \leq a_R - \kappa r_R^2 \leq C < \infty.$$

If $s_R = 0$ and a fixed response floor $\kappa > 0$ survives, then

$$\eta_R^2 = O(n_R), \quad \eta_R = O(\sqrt{n_R}).$$

If one also has $a_R - \kappa r_R^2 \geq c > 0$, the active block retains a nonvanishing reserve while the mixing still obeys the same soft scaling.

Proof. From [\(14\)](#),

$$\eta_R^2 \leq (a_R - \kappa r_R^2) n_R \leq C n_R.$$

$\square$

7. DIRICHLET ACTIVATION OF CHARACTER RESPONSE

The charge envelope still knows no energy. The missing arrow is the passage form. In the selected pure- $SU(3)$ confinement problem [\[7\]](#), the charged cut lift satisfies the source-twisted jump and fails ordinary admissible gluing. Friedrichs restriction measures that failure with the local physical form. That is where the character response becomes energetic.

Let $\ell \in A_{\text{uns}} \setminus \{0\}$. Fix a certified family of detecting characters and assume their selected cut lifts land in one common boundary form domain $\mathcal{D}_\partial$ carrying a single nonnegative ordinary Friedrichs/Dirichlet form d^{adm} . For every detecting $\alpha \in \hat{A}_{\text{uns}}$ and every u in the declared common source domain, suppose the boundary trace $T_{\ell,\alpha} u \in \mathcal{D}_\partial$ and the ordinary seam defect factors as

$$(16) \quad \beta_{\ell,\alpha}(u) = (\alpha(\ell) - 1) T_{\ell,\alpha} u.$$

Proposition 7.1 (Character-factor Dirichlet activation). *Under (16),*

$$(17) \quad d^{\text{adm}}[\beta_{\ell,\alpha}(u)] = |\alpha(\ell) - 1|^2 d^{\text{adm}}[T_{\ell,\alpha}u].$$

If, on the certified family under consideration,

$$(18) \quad d^{\text{adm}}[T_{\ell,\alpha}u] \geq c_* \|u\|^2, \quad c_* > 0,$$

then

$$(19) \quad d^{\text{adm}}[\beta_{\ell,\alpha}(u)] \geq c_* r_\ell(\alpha) \|u\|^2.$$

Consequently, for torsion charge of order m ,

$$(20) \quad \inf_{\alpha(\ell) \neq 1} d^{\text{adm}}[\beta_{\ell,\alpha}(u)] \geq 4c_* \sin^2 \frac{\pi}{m} \|u\|^2.$$

Proof. The boundary form is quadratic, so scalar multiplication in (16) gives (17). The remaining statements follow from (18) and [theorem 4.1](#). $\square$

Remark 7.2 (What torsion does not prove). The factor $4\sin^2(\pi/m)$ is already exact, but it is only the charge-side reserve. The Hamiltonian does not know it until the trace reaches the Dirichlet form with a positive bound. Thus (18), or the sharper finite passage-pencil certificate, remains load-bearing. In the present pure- $SU(3)$ confinement manuscript that last normalized pencil calculation is still open [7].

Remark 7.3 (Associator covariance). This is where the associator belongs. On the selected $SU(3)$ passage pencil it acts by unitary reassociation, so changing fusion presentation does not change the generalized spectrum. The center charge supplies the defect in (16). The associator keeps the comparison coherent. It does not create the passage energy.

8. SECONDARY COHERENCE-SENSITIVE ANALYTIC ACTIVATION

The theorem above needs no direct H^3 -to-energy map. There is still another possibility: a coherence-sensitive comparison may acquire its own analytic representative. If that happens, it has to be typed separately.

Definition 8.1 (Analytic activation witness). Let χ be a coherence-sensitive physical comparison in a fixed finite certified realization. An *analytic activation witness* for χ is a separately certified sesquilinear mixed form

$$h^\chi : \mathcal{D}_A \times \mathcal{D}_S \rightarrow \mathbb{C}$$

whose carrier, support, complete ground, boundary data, refinement stage, and provenance agree with those of χ , and which satisfies the declared boundedness or form-boundedness needed for a finite represented mixed block B^χ .

No mixed block is inferred merely from a nontrivial associator, transgressed selector, or calibrated mixed phase. This preserves the coordinate firewall imported from [8].

8.1. Pure activation from zero baseline mixing. Fix finite positive diagonal blocks and a represented activation block B . Let

$$K_\tau = \begin{pmatrix} A & \tau B^* \\ \tau B & N \end{pmatrix}, \quad W = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau \geq 0.$$

Here τ is an analytic activation strength, not a norm of $[\omega]$.

Theorem 8.2 (Pure activation threshold). *Assume $A > 0$, $R > 0$, $N \geq 0$, and $\text{Ker } N \subseteq \text{Ker } B^*$. Put*

$$C = B^* N^\dagger B.$$

If $B \neq 0$, define

$$(21) \quad \tau_c^{-2} = \left\| A^{-1/2} C A^{-1/2} \right\|;$$

if $B = 0$, set $\tau_c = \infty$. Then $K_\tau \geq 0$ exactly for $0 \leq \tau \leq \tau_c$. On that interval the response floor is

$$(22) \quad \kappa_{\text{aff}}(\tau) = \lambda_{\min} \left[R^{-1/2} (A - \tau^2 C) R^{-1/2} \right].$$

Moreover $\kappa_{\text{aff}}(\tau)$ is nonincreasing in τ^2 , and for $0 \leq \sigma, \tau \leq \tau_c$,

$$(23) \quad |\kappa_{\text{aff}}(\tau) - \kappa_{\text{aff}}(\sigma)| \leq |\tau^2 - \sigma^2| \left\| R^{-1/2} C R^{-1/2} \right\|.$$

For $B \neq 0$, $\kappa_{\text{aff}}(\tau) > 0$ for $0 \leq \tau < \tau_c$ and $\kappa_{\text{aff}}(\tau_c) = 0$.

Proof. By [theorem 6.3](#) with $R_S = 0$, positivity of K_τ is equivalent to

$$A - \tau^2 C \geq 0.$$

Conjugating by $A^{-1/2}$ gives $I - \tau^2 A^{-1/2} C A^{-1/2} \geq 0$, which is equivalent to $0 \leq \tau \leq \tau_c$. On this positivity interval, $K_\tau \geq \kappa W$ is equivalent to

$$A - \kappa R - \tau^2 C \geq 0.$$

Conjugation by $R^{-1/2}$ gives (22). Since $C \geq 0$, increasing τ^2 decreases the Hermitian matrix in form order, proving monotonicity of its least eigenvalue. The estimate (23) is the standard finite-dimensional least-eigenvalue perturbation bound; compare [4]. Strict positivity below τ_c and vanishing at τ_c follow from the definition of the threshold. $\square$

Remark 8.3 (Nonzero baseline mixing). If $B_\tau = B_0 + \tau C$, the first variation of the Schur complement contains a linear interference term. The monotonicity assertion of [theorem 8.2](#) is therefore restricted to pure activation from vanishing baseline mixing with fixed diagonal blocks.

9. APPLICATION III: FREE MAXWELL CONTROL

Maxwell is the control problem. If response coercivity were merely a mass gap written in softer notation, it should fail here. It does not. We use the standard free bosonic Fock realization [2] and the usual Weyl criterion [12].

9.1. One-photon representation. Use the physical helicity one-photon space

$$\mathcal{H}_1 = L^2(\mathbb{R}^3, d^3k; \mathbb{C}^2),$$

with

$$(Hf)(k) = |k|f(k), \quad (\mathbf{P}f)(k) = kf(k).$$

Thus the joint four-momentum is $p(k) = (|k|, k)$, lying on the future null cone away from $k = 0$.

Let $u = (u^0, \mathbf{u})$ be a future unit timelike observer, and define

$$H_u = u^0 H - \mathbf{u} \cdot \mathbf{P}.$$

If u has rapidity ρ , then for every k

$$e^{-\rho}|k| \leq u^0|k| - \mathbf{u} \cdot k \leq e^{\rho}|k|,$$

so in operator form

$$(24) \quad e^{-\rho}H \leq H_u \leq e^{\rho}H.$$

Choose four future-timelike observers $u_1, \dots, u_4$ spanning $\mathbb{R}^{1,3}$ and weights $w_j > 0$. Define

$$(25) \quad \mathfrak{r}_{\text{aff}}[f] = \sum_{j=1}^4 w_j \langle f, H_{u_j} f \rangle.$$

The spanning condition makes the finite response packet injective on four-momentum. From (24), with

$$c_{\text{aff}} = \sum_j w_j e^{-\rho_j}, \quad C_{\text{aff}} = \sum_j w_j e^{\rho_j},$$

we obtain

$$(26) \quad c_{\text{aff}} \langle f, Hf \rangle \leq \mathfrak{r}_{\text{aff}}[f] \leq C_{\text{aff}} \langle f, Hf \rangle.$$

Hence

$$(27) \quad \langle f, Hf \rangle \geq C_{\text{aff}}^{-1} \mathfrak{r}_{\text{aff}}[f].$$

Proposition 9.1 (Maxwell gaplessness with positive response floor). *Let $f \in \mathcal{H}_1$ be normalized, smooth, and supported in $1 \leq |k| \leq 2$. Set $R_n = 3^n$ and*

$$f_n(k) = R_n^{3/2} f(R_n k).$$

Then $\|f_n\| = 1$, the supports are pairwise disjoint, $f_n \rightharpoonup 0$, and

$$\|Hf_n\| = R_n^{-1} \|q|f(q)\| \longrightarrow 0.$$

Thus f_n is a Weyl sequence at spectral value 0 for the one-photon energy operator. The corresponding one-photon Fock states $\Psi_n = a^\dagger(f_n)\Omega$ are normalized, vacuum-orthogonal, weakly null, and satisfy $\|H\Psi_n\| \rightarrow 0$.

Nevertheless the response inequality (27) holds on the natural one-photon form domain. Consequently

$$\Delta_{\text{Maxwell}} = 0, \quad \kappa_{\text{aff}} \geq C_{\text{aff}}^{-1} > 0.$$

Proof. Normalization follows by the change of variables $q = R_n k$. Because the radial support of f_n is contained in $[R_n^{-1}, 2R_n^{-1}]$ and $R_{n+1} = 3R_n$, the supports are disjoint. Orthogonality therefore implies weak convergence to zero.

Moreover

$$\|Hf_n\|^2 = \int |k|^2 R_n^3 |f(R_n k)|^2 d^3 k = R_n^{-2} \int |q|^2 |f(q)|^2 d^3 q.$$

The Weyl criterion then places 0 in the one-photon spectrum [12]. Second quantization preserves the one-particle action on one-photon vectors. Finally (27) provides the positive response floor independently of the absence of a Hilbert-space gap. $\square$

Corollary 9.2 (Infrared response softness). *For the family in theorem 9.1,*

$$\mathfrak{r}_{\text{aff}}[f_n] = O(R_n^{-1}) \longrightarrow 0.$$

Remark 9.3 (Continuous center is not the proof). The masslessness used here comes from the Maxwell dynamics. The fact that $Z(U(1)) = U(1)$ is not a substitute for that calculation. The charge-side softness and the photon-side softness meet in this example; neither proves the other in general.

10. REDUCTIVE GAPPED/SOFT COEXISTENCE

The next question is whether the two behaviors can live on one selected branch. They can, but the mixing has to pay for itself. We formulate the finite or finite-reduced problem for

$$G = \frac{G_{\text{ss}} \times \mathbb{T}^k}{\Gamma},$$

with Γ finite central. The local Lie algebra splits into semisimple and toral parts, but the quotient modifies the genuine global charge and cocharacter lattices; standard compact Lie-group background may be found in [3, 5].

Let P_Ω be the complete-ground projection and work on a selected ground-orthogonal excitation space

$$\mathcal{V} = (I - P_\Omega)\mathcal{H} = \mathcal{V}_{\text{ss}} \oplus \mathcal{V}_{\text{tor}}.$$

Assume on this finite or finite-reduced endpoint that

$$K = \begin{pmatrix} A & B^* \\ B & N \end{pmatrix}, \quad W = \begin{pmatrix} R_{\text{ss}} & 0 \\ 0 & R_{\text{tor}} \end{pmatrix}.$$

Hypothesis 10.1 (Pure-toral infrared admissibility). *There exists a normalized infrared trial family y_j in a selected toral form domain such that*

$$(0, y_j) \in \mathcal{D}(K) \subseteq (I - P_\Omega)\mathcal{H}$$

after all selected Gauss/Ward and physical-carrier reductions, and

$$N[y_j] \rightarrow 0.$$

Theorem 10.2 (Reductive coexistence). *Assume*

$$A \geq \kappa_{\text{ss}} R_{\text{ss}}, \quad N \geq \kappa_{\text{tor}} R_{\text{tor}}, \quad \kappa_{\text{ss}}, \kappa_{\text{tor}} > 0.$$

Fix

$$0 < \kappa < \min\{\kappa_{\text{ss}}, \kappa_{\text{tor}}\}$$

and define $N_\kappa = N - \kappa R_{\text{tor}}$. If

$$N_\kappa \geq 0, \quad \text{Ker } N_\kappa \subseteq \text{Ker } B^*, \quad B^* N_\kappa^\dagger B \leq A - \kappa R_{\text{ss}},$$

then

$$K \geq \kappa W.$$

Under [theorem 10.1](#), however,

$$K[0 \oplus y_j] = N[y_j] \rightarrow 0.$$

Thus the full nonvacuum form infimum is zero even though the response floor stays positive.

Proof. The response estimate is precisely [theorem 6.3](#). The pure-toral trial family lies in the full selected form domain by [theorem 10.1](#); hence its form energy tends to zero. A positive Hilbert-norm lower bound on the entire nonvacuum form domain is therefore impossible. Moreover $N[y_j] \geq \kappa_{\text{tor}} R_{\text{tor}}[y_j]$ implies $R_{\text{tor}}[y_j] \rightarrow 0$. $\square$

Remark 10.3 (Rayleigh versus Weyl). In an abstract form setting, a normalized family with $q[\psi_j] \rightarrow 0$ will be called an *infrared Rayleigh family*. It is promoted to a Weyl sequence only when the appropriate operator-domain criterion, as in [theorem 9.1](#), is separately verified.

11. RESPONSE-FLOOR LIMIT TRANSFER

Nothing in the finite Schur estimate gives the limit for free. Mosco convergence and its varying-Hilbert-space variants provide the analytic passage [\[6, 11\]](#); the source-specific comparison maps come from [\[9\]](#). The result below begins only after those hypotheses are in place. It is a transfer lemma, not a replacement for the upstream comparison–Mosco theorem.

Hypothesis 11.1 (Joint form/response recovery). *Assume $q_n \rightarrow q_0$ in an independently certified comparison–Mosco sense. Let $\mathfrak{r}_n$ and $\mathfrak{r}_0$ be nonnegative response forms. For every $u \in \mathcal{D}(q_0)$, assume there exists a recovery family u_n such that*

$$q_n[u_n] \rightarrow q_0[u], \quad \mathfrak{r}_n[u_n] \rightarrow \mathfrak{r}_0[u].$$

Lemma 11.2 (Response-floor transfer). *Under [theorem 11.1](#), suppose one constant $\kappa_* > 0$ satisfies*

$$q_n[v] \geq \kappa_* \mathfrak{r}_n[v]$$

for every n and every $v \in \mathcal{D}(q_n)$. Then

$$q_0[u] \geq \kappa_* \mathfrak{r}_0[u]$$

for every $u \in \mathcal{D}(q_0)$.

Proof. Apply the finite inequality to a joint recovery family and pass to the limit. $\square$

Corollary 11.3 (Limiting response softness). *If $q_0[u_j] \rightarrow 0$, then $\mathfrak{r}_0[u_j] \rightarrow 0$.*

Remark 11.4 (What is not proved here). The hard part is joint recovery of the response on the moving carrier. [Theorem 11.2](#) starts after that bridge has been built. In particular, the response-weighted floor cannot simply be substituted for a full off-ground hypothesis inside an existing comparison–Mosco theorem and expected to carry the rest of the proof with it.

12. PERSISTENT COHERENCE-ACTIVATION RESERVE

Once a coherence-sensitive mixed form has been represented, refinement creates a new danger: the finite activation may be harmless at each early stage and still exhaust the reserve cofinally. At each endpoint D take a certified pure-activation package (A_D, B_D, N_D, R_D) satisfying the finite hypotheses of [theorem 8.2](#): $A_D > 0$, $R_D > 0$, $N_D \geq 0$, and $\text{Ker } N_D \subseteq \text{Ker } B_D^*$. Fix $\kappa > 0$ as the floor to be preserved, put

$$C_D = B_D^* N_D^\dagger B_D,$$

and assume $A_D - \kappa R_D > 0$.

Definition 12.1 (Coherence-activation load). Define

$$(28) \quad \mathfrak{M}_{\text{coh}}(D; \kappa) = \left\| (A_D - \kappa R_D)^{-1/2} C_D (A_D - \kappa R_D)^{-1/2} \right\|.$$

Proposition 12.2 (Persistent activation capacity). *If*

$$\sup_D \mathfrak{M}_{\text{coh}}(D; \kappa) = M_* < \infty,$$

then, when $M_* > 0$, every activation strength

$$0 \leq \tau < M_*^{-1/2}$$

preserves the response floor κ at every finite endpoint. If $M_* = 0$, every finite τ preserves the floor. If the branch also satisfies [theorem 11.1](#) for the activated forms and responses, the same response floor passes to the limiting response form.

Proof. The inequality $\tau^2 \mathfrak{M}_{\text{coh}}(D; \kappa) < 1$ is exactly the conjugated Schur condition

$$A_D - \kappa R_D - \tau^2 C_D > 0.$$

Uniform boundedness gives the same admissible τ at every endpoint. The limit statement is [theorem 11.2](#). $\square$

Corollary 12.3 (Refinement obstruction). *If $\mathfrak{M}_{\text{coh}}(D_j; \kappa) \rightarrow \infty$ along a cofinal sequence, then, within the pure-activation family above, no fixed nonzero activation strength preserves the floor κ uniformly along that sequence.*

13. EFFECTIVE INFRARED RESTORING FORM

There is a different question again. Response coercivity asks whether a chosen response keeps a reserve. Symmetry breaking asks which candidate gauge directions remain null in the infrared restoring problem. We therefore change objects. The notation below is deliberately *restoring form*, not mass-squared matrix. A physical mass interpretation still requires a model-specific kinetic normalization and zero-momentum identification.

Let $\mathcal{V}_R$ and $\mathcal{V}_C$ be finite-dimensional real inner-product spaces and put

$$\mathcal{V} = \mathcal{V}_R \oplus \mathcal{V}_C, \quad Q_{\text{rest}} = \begin{pmatrix} A & B^* \\ B & N \end{pmatrix} \geq 0, \quad A > 0.$$

The same Schur statement holds for Hermitian blocks after passage to the underlying real space. At this stage $\mathcal{V}_C$ is only an analytic candidate space; no identification with the gauge Lie algebra is being made.

Theorem 13.1 (Effective restoring form). *Define*

$$(29) \quad Q_{\text{rest}}^{\text{eff}} = N - BA^{-1}B^*.$$

Then $Q_{\text{rest}}^{\text{eff}} \geq 0$ and

$$\text{Ker } Q_{\text{rest}} \cong \text{Ker } Q_{\text{rest}}^{\text{eff}}.$$

The correspondence sends

$$y \mapsto \begin{pmatrix} -A^{-1}B^*y \\ y \end{pmatrix}.$$

Proof. Complete the square:

$$\begin{aligned} Q_{\text{rest}}[x \oplus y] &= \left\langle x + A^{-1}B^*y, A(x + A^{-1}B^*y) \right\rangle \\ &\quad + \left\langle y, (N - BA^{-1}B^*)y \right\rangle. \end{aligned}$$

The first term is nonnegative. Setting $x = -A^{-1}B^*y$ and using $Q_{\text{rest}} \geq 0$ gives

$$\langle y, Q_{\text{rest}}^{\text{eff}}y \rangle \geq 0,$$

so $Q_{\text{rest}}^{\text{eff}} \geq 0$. The displayed decomposition is now a sum of two nonnegative terms. It vanishes exactly when $x = -A^{-1}B^*y$ and $y \in \text{Ker } Q_{\text{rest}}^{\text{eff}}$, which gives the stated kernel correspondence. $\square$

Remark 13.2 (Candidate nullity only). A kernel is not yet a particle statement. The nullity of $Q_{\text{rest}}^{\text{eff}}$ counts candidate infrared-restoring directions. To call one a massless gauge direction still requires the Ward/Gauss and kinetic certificates.

14. PERSISTENT CANDIDATE-SOFT PROJECTORS

Let $s \mapsto Q_{\text{rest}}^{\text{eff}}(s)$ be a norm-continuous finite-dimensional family of non-negative self-adjoint operators. Spectral projections around isolated spectral clusters are standard in perturbation theory [4].

Proposition 14.1 (Persistent candidate-soft projector). *Let I be a nonempty interval and choose $s_0 \in I$. Put*

$$r = \dim \text{Ker } Q_{\text{rest}}^{\text{eff}}(s_0).$$

Assume that one $\delta > 0$ satisfies

$$\text{spec}(Q_{\text{rest}}^{\text{eff}}(s)) \setminus \{0\} \subseteq [\delta, \infty) \quad (s \in I).$$

Choose the fixed positively oriented circle

$$\Gamma_0 = \{z \in \mathbb{C} : |z| = \delta/2\}$$

and set

$$P_{\text{soft}}(s) = \frac{1}{2\pi i} \oint_{\Gamma_0} (z - Q_{\text{rest}}^{\text{eff}}(s))^{-1} dz.$$

Then $P_{\text{soft}}(s)$ varies norm-continuously on I and

$$\text{rank } P_{\text{soft}}(s) = r \quad (s \in I).$$

Proof. The uniform spectral gap places the fixed circle Γ_0 in the resolvent set of every $Q_{\text{rest}}^{\text{eff}}(s)$. Norm continuity of the family and the resolvent identity give norm continuity of the resolvent on Γ_0 , hence of the contour integral. The rank of a norm-continuous projection-valued map is locally constant. Since I is connected and the rank at s_0 is r , the rank is r throughout I ; see [4]. $\square$

15. GLOBAL COMPLETION OF THE SOFT GAUGE SECTOR

A kernel is not a gauge group. The restoring null space is still analytic data. To obtain a genuine residual compact subgroup we have to put back the symmetry certificate and the global form, separately.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{h}_{\text{Ward}} \subseteq \mathfrak{g}$ be the certified real Lie subalgebra of infinitesimal gauge directions preserving the required Gauss/Ward package. The analytic candidate space $\mathcal{V}_C$ and the gauge Lie algebra are different types, so introduce a specified injective real-linear gauge-realization map

$$j_{\mathfrak{g}} : \mathcal{V}_C \longrightarrow \mathfrak{g}.$$

Define the Ward-compatible soft algebra by

$$\mathfrak{g}_{\text{soft}} = j_{\mathfrak{g}}(\text{Ker } Q_{\text{rest}}^{\text{eff}}) \cap \mathfrak{h}_{\text{Ward}}.$$

Assume explicitly that $\mathfrak{g}_{\text{soft}}$ is contained in the Lie algebra $\mathfrak{t}_G$ of a maximal torus $T_G \subseteq G$; since $\mathfrak{g}_{\text{soft}}$ is a real vector subspace of $\mathfrak{g}$ and $\mathfrak{t}_G$ is Abelian, it is then an Abelian Lie subalgebra. For a compact connected Lie group, the

kernel of the exponential map on a maximal torus is a full lattice; standard references include [3, 5]. Define

$$\Lambda_G = \text{Ker}(\exp : \mathfrak{t}_G \rightarrow T_G).$$

Definition 15.1 (Rational soft algebra). The soft algebra is *rational* when

$$\text{rank}_{\mathbb{Z}}(\mathfrak{g}_{\text{soft}} \cap \Lambda_G) = \dim_{\mathbb{R}} \mathfrak{g}_{\text{soft}}.$$

Theorem 15.2 (Global soft-subgroup completion). *If the realized Ward-compatible soft algebra is torally contained and rational with respect to Λ_G , then*

$$H_{\text{soft}}^0 = \exp(\mathfrak{g}_{\text{soft}}) \cong U(1)^r, \quad r = \dim_{\mathbb{R}} \mathfrak{g}_{\text{soft}}.$$

For $r = 1$, the surviving connected compact subgroup is a circle. The primitive lattice element in $\mathfrak{g}_{\text{soft}} \cap \Lambda_G$ fixes its natural global generator normalization.

Proof. Rationality means that $\mathfrak{g}_{\text{soft}} \cap \Lambda_G$ is a full lattice in the r -dimensional real vector space $\mathfrak{g}_{\text{soft}}$. Hence

$$\mathfrak{g}_{\text{soft}} / (\mathfrak{g}_{\text{soft}} \cap \Lambda_G)$$

is a compact r -torus. Its image under the exponential map is exactly the connected subgroup generated by $\mathfrak{g}_{\text{soft}}$, and therefore isomorphic to $U(1)^r$. Compare the compact-torus structure in [3, 5]. $\square$

Remark 15.3 (Mixed generators). A surviving generator need not lie entirely in an original $U(1)$ factor. It may contain both a Cartan component of a semisimple factor and an original toral component. The restoring Schur complement determines an analytic candidate direction, the realization map j_g identifies the corresponding gauge generator, and the global quotient and lattice determine whether that generator closes to a circle or higher torus.

16. FIREWALLS AND EXACT STATUS BOUNDARY

These are not disclaimers. They are theorem boundaries. Crossing any one of them would change the claim of the paper.

Firewall 16.1 (Associator is not a mass gap).

$$[\omega] \neq 0 \quad \not\Rightarrow \quad \Delta > 0.$$

Firewall 16.2 (Finite center is not a gap theorem).

$$|Z(G)| < \infty \quad \not\Rightarrow \quad \Delta > 0.$$

Firewall 16.3 (Continuous center is not a masslessness theorem).

$$Z(G)^0 \neq \{e\} \quad \not\Rightarrow \quad \Delta = 0.$$

The Maxwell theorem is a statement about a concrete dynamical realization, not a consequence of the abstract group alone.

Firewall 16.4 (Response coercivity is not full spectral coercivity).

$$\kappa_{\text{aff}} > 0 \quad \not\Rightarrow \quad \Delta > 0.$$

[Theorem 9.1](#) proves the failure of this converse.

Firewall 16.5 (Affine observer response is not full spacetime reconstruction). The response form used here is built from observer-energy data $p \cdot u$. A theorem that this response-weighted coercivity reconstructs local incidence, localization, or a complete affine spacetime requires an additional interface beyond the scope of this paper.

Firewall 16.6 (Restoring nullity is not automatically unbroken gauge symmetry). A vector in $\text{Ker } Q_{\text{rest}}^{\text{eff}}$ is only a candidate soft direction until it passes the Ward/Gauss certificate and the global torus-lattice closure test.

Firewall 16.7 (Restoring form is not automatically a mass matrix). The notation M^2 may be used only after a model-specific physical mass interpretation has been independently certified.

Firewall 16.8 (No electroweak derivation). A one-dimensional mixed Cartan/toral soft algebra can mathematically integrate to a residual $U(1)$. No Standard Model Higgs mechanism, Weinberg angle, coupling relation, or particle-mass formula is derived here.

17. INTERPRETATION

The two envelopes answer the same formal question in different mathematics:

$$\rho_{\text{obs}}(p) = \inf_u p \cdot u = \sqrt{p^2}, \quad \rho_{\text{ch}}(\ell) = \inf_{\alpha(\ell) \neq 1} |\alpha(\ell) - 1|^2.$$

Timelike versus null momentum is spectral geometry. Torsion versus infinite order is charge arithmetic. The formulas rhyme. The physics does not collapse with them.

The place where physics re-enters is the analytic bridge. A finite form may dominate a response, or it may not. The Schur complement measures the reserve left after soft-sector mixing. The Dirichlet passage gives the charge-side realization. Maxwell gives the opposite control: the full gap can disappear while the observer response remains well typed and positively controlled.

The associator stays upstream unless an independent analytic representative is actually built. That is the right place to look for a coherence effect. Not before.

So the proof can be compressed to

$\begin{array}{c} \text{comparison} \rightarrow \text{response} \rightarrow \text{lower envelope} \\[10pt] \frac{\text{analytic bridge}}{\text{separate certificate}} \rightarrow \text{energy reserve.} \end{array}$

A mass gap is one strong realization of that pattern. It is not the definition.

18. CONCLUSION

A mass gap says that every nonvacuum direction pays. Lower-envelope rigidity asks less. It asks whether a particular certified response can be pushed into degeneracy. Timelike momentum and torsion charge carry positive reserves; null momentum and infinite-order charge do not. Those are different physical statements with the same formal shape.

Energy is another step. The Schur problem tells us when a physical form controls a response in the presence of soft directions. The Dirichlet passage realizes that step for the selected charge problem. Maxwell shows why the distinction matters: its gap vanishes, but the affine observer response does not become meaningless.

Everything downstream keeps the same discipline. A restoring kernel is not yet a gauge group. An associator is not a mass gap. A finite charge reserve is not a confinement theorem. The useful statement is the narrower one:

rigidity is separation of certified response from degeneracy;
energy is another theorem.

That is enough.

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INDEPENDENT RESEARCHER, COLUMBUS, OHIO, USA